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Spectrum-generating functions for strings and superstrings

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Abstract. The spectrum-generating functions for a variety of strings and superstrings are derived. These are expressed in terms of characters of irreducible representations of the appropriate transverse spacetime symmetry group and, in the case of heterotic strings, the relevant gauge group $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2$. The expansions involve principal specialisations of characters of both GL(N) and GL(M/N) in the limit as M and N tend to infinity. By using Schur function methods, modification rules and various theta function identities a number of spectrum-generating functions are derived and tabulated. The equivalence of the GSO-projected Neveu-Schwarz-Ramond superstring and the Green-Schwarz superstring is proved, and the connection between the lattice formulation and the GSO-projected Neveu-Schwarz-Ramond formulation of heterotic strings is spelled out.

1. Introduction

Various group theoretical aspects of a number of different string models have already been explored and exploited. In particular Ramond [1] has described a procedure for determining explicitly the transformation properties of the string states of given mass under the action of the group SO($D-1$) relevant to strings in D-dimensional spacetime. His work covered not only bosonic strings but also fermionic strings and indeed heterotic strings. However, for all but low-lying mass states the method used was prone to calculational complexities which concealed possible regularities in the string spectra and gave little hope of determining the degeneracies of particular irreducible representations at any given level.

A different approach which is well suited to solving this latter problem has recently been enunciated by Curtright and Thorn [2]. This depends on writing down for each string a spectrum-generating function $\chi(x, q)$ in the form of an SO($D-1$) character generator. When expanded in the form

$$\chi(x, q) = q^{L_0} \sum_{L=0}^{\infty} \sum_{\lambda} n_L^\lambda \chi^\lambda(x)_{D-1} q^L \tag{1.1}$$

the coefficient n_L^λ gives the number of times the irreducible representation λ of SO($D-1$) having character $\chi^\lambda(x)_{D-1}$ occurs in the level L. In flat spacetime the mass squared of this level is $L_0 + L$, where L_0 is the mass squared of the vacuum state.

In D-dimensional spacetime string theories the relevant transverse symmetry group is SO($D-2$). Massless states span representations of this group whilst the massive states span representations of the larger group SO($D-1$). In the cases under consideration D is even and it is convenient, as in (1.1), to express group characters in terms of the parameters $x=(x_1, x_2, \ldots, x_{2k-1}, x_{2k}, 1)$ with $D-1=2k+1$, where the components of x are eigenvalues of an arbitrary $SO(2k+1)$ group element. In accordance with the usual conjugacy class parametrisation of $SO(2k+1)$ we have $x=(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_k}, e^{-i\phi_1}, e^{-i\phi_2}, \ldots, e^{-i\phi_k}, 1)$. The restriction to $SO(2k)$ is effected as far as the parametrisation of characters is concerned merely by dropping the last component 1 from x. No confusion should result from using the same symbol x in connection with both $SO(2k)$ and $SO(2k+1)$. It should further be stressed that the components have been arranged so that

$$x_{k+j}=x_j^{-1}=e^{-i\phi_j} \qquad j=1,2,\ldots,k. \tag{1.2}$$

This pattern will recur throughout this paper.

For a number of string models it is a very straightforward matter to write down the appropriate spectrum-generating function and the only task remaining is that of expanding it in the form (1.1). Curtright *et al* [3], guided by an earlier conjecture [2], have succeeded in carrying out this expansion explicitly in the case of the open bosonic string. The results involve the principal specialisation of characters of $GL(N)$ as $N\to\infty$.

Subsequently this work was extended in [4] to the case of the open Neveu-Schwarz [5] and Ramond [6] strings which each possess both bosonic and fermionic excitation operators. In these two cases the relevant expansions (1.1) involve two specialisations of characters of the supergroup $GL(M/N)$ with both $M\to\infty$ and $N\to\infty$.

In the present paper it is demonstrated that these results may all be obtained very easily by exploiting S-function methods as expounded first by Littlewood [7] and to some extent summarised by Macdonald [8]. These techniques shed light on a number of aspects of the problem: on the way that characters of supergroups enter the analysis; on the identification of certain prefactors, appropriate to the $k\to\infty$ limit, with infinite series of S functions and their generalisation to infinite series of supersymmetric functions; and on the necessity of applying the modification rules of $SO(2k+1)$ to correct for the fact that k is actually finite. In this way it has been possible to deal with each of the three open string models in a unified manner including, for the first time, the complete analysis of the spinor characters of $SO(2k+1)$ which necessarily arise in the case of the Ramond string.

The paper is organised so that the open bosonic string, the Neveu-Schwarz string and the Ramond string are dealt with successively in §§ 2, 3 and 4, respectively.

Although they each contain both bosonic and fermionic excitation operators neither the Neveu-Schwarz model nor the Ramond model is supersymmetric and the former contains a tachyonic ground state. A very considerable advance was made by Gliozzi *et al* [9] who applied a projection which restricted the Neveu-Schwarz model to states of even G-parity and the Ramond model to one for which the ground state was a Majorana-Weyl spinor. By setting the number of spacetime dimensions to ten, they then showed that the number of physical states of the bosonic Neveu-Schwarz sector and the fermionic Ramond sector are equal at each mass level. They did this by exploiting an abstruse identity due to Jacobi [10, p 147].

Inspired by this development Green and Schwarz [11] produced a manifestly supersymmetric version of this model based on the same ground state as the ten-dimensional GSO-projected Neveu-Schwarz-Ramond model but incorporating different excitation operators.

The equivalence of these models, which is not at all obvious, is discussed in § 5. The GSO-projected Neveu-Schwarz-Ramond spectrum-generating function is written down and is then shown to be identical to the Green-Schwarz spectrum-generating function. This result is established by making use of a theta function identity, again due to Jacobi [10, p 507], which seems to have been exploited first in this context by Nahm [12]. A particular merit of this identification of the GSO-projected Neveu-Schwarz-Ramond model and the Green-Schwarz model is that the spectrum of the latter is then determined to be that of the former. The first few levels are then tabulated on the basis of the calculations of §§ 2 and 3.

Section 6 is concerned with the heterotic string models of Gross *et al* [13] involving the gauge groups $E_8 \times E_8$ and $\mathrm{Spin}(32)/Z_2$. These models involve certain even self-dual lattices associated with the weight spaces of the gauge groups. By making use of theta function expansions, yet again due to Jacobi [10, p 501], the gauge group contributions to the spectrum-generating function are rewritten in the form of factors of the type appearing in the GSO-projected Neveu-Schwarz-Ramond model. In this way they are explicitly evaluated and tabulated. This enables the complete spectrum-generating function of each of the heterotic string models to be written down. Of course, these models are closed string models with left- and right-hand sectors so that the spectrum-generating functions take the form

$$\chi(x, t, q, r) = \chi^{\mathrm{R}}(x, r)\chi^{\mathrm{L}}(x, t, q) \tag{1.3}$$

with

$$\chi^{\mathrm{R}}(x, r) = r^{R_0} \sum_{R=0}^{\infty} \sum_{\lambda} m_R^{\lambda} \chi^{\lambda}(\boldsymbol{x})_{D-1} r^R \tag{1.4}$$

exactly as in (1.1), but with

$$\chi^{\mathrm{L}}(x, t, q) = q^{L_0} \sum_{L=0}^{\infty} \sum_{\mu,\nu} n_L^{\mu,\nu} \chi^{\mu}(\boldsymbol{x})_{D-1} \chi^{\nu}(\boldsymbol{t})_{\mathrm{G}} q^L. \tag{1.5}$$

Here $\chi^{\mu}(\boldsymbol{x})_{D-1}$ and $\chi^{\nu}(\boldsymbol{t})_{\mathrm{G}}$ denote characters of irreducible representations of the spacetime symmetry group $\mathrm{SO}(D-1)$ and the gauge group G, respectively. The boundary conditions for closed string models imply that the levels of the right- and left-hand sectors must be matched. Thus in the expansion of (1.3), since (1.3) is the product of (1.4) and (1.5), it is only necessary to retain those terms for which

$$R_0 + R = L_0 + L. \tag{1.6}$$

This matching procedure is applied to the heterotic string models to produce a table indicating both the spacetime SO(9) content and the gauge group content of the ground state and the first five excited states.

Some concluding remarks are made in § 7 which emphasise that the techniques used here have a wide range of applicability, extending for example to the $D = 26$ giant superstring model of Thierry-Mieg [14].

2. The open bosonic string

In the case of the open bosonic string the vacuum state is an $\mathrm{SO}(2k)$ singlet state which is tachyonic with mass squared $L_0 = -1$ whilst excited states are generated by the action of the bosonic operators a_{-n}^i with $i = 1, 2, \ldots, 2k$ and $n = 1, 2, \ldots, \infty$. These operators, for each fixed n, transform as the basis states of the defining vector $2k$-dimensional irreducible representation of $\mathrm{SO}(2k)$. Moreover each operator a_{-n}^i contributes n to the mass squared value. It follows that the required spectrum-generating function (1.1) is given by

$$\chi_{\mathrm{B}}(\boldsymbol{x}, q)=q^{-1} \prod_{n=1}^{\infty} \prod_{i=1}^{2k} (1-x_i q^n)^{-1}. \tag{2.1}$$

The factor $x_i q^n$ is associated with the operator a^i_{-n}, x_i takes care of the SO($2k$) transformation properties and q^n takes care of the mass squared contribution. The inverse power of each factor in the product allows for the fact that the operators are indeed bosonic so that multiple excitations may occur.

The key to expanding this generating function is the formula [7, p 103; 8, p 33]

$$\prod_{n=1}^{N} \prod_{i=1}^{2k} (1-x_i y_n)=\sum_{\sigma} \{\sigma\}(\boldsymbol{y})_N \{\sigma\}(\boldsymbol{x})_{2k} \tag{2.2}$$

where the summation is carried out over all partitions $\sigma=(\sigma_1, \sigma_2, \ldots)$. Such a partition specifies both an irreducible representation of GL(N) with character $\{\sigma\}(\boldsymbol{y})_N$ and an irreducible representation of GL($2k$) with character $\{\sigma\}(\boldsymbol{x})_{2k}$. These characters are symmetric functions of their arguments and are known as Schur functions or S functions [7], which may also be denoted [8] by $s_\sigma(\boldsymbol{y})$ and $s_\sigma(\boldsymbol{x})$, as appropriate.

What is really required in applying (2.3) to (2.2) is the principal specialisation of the character $\{\sigma\}(\boldsymbol{y})_N$ for which $\boldsymbol{y}=\boldsymbol{q}$ with $q_n=q^n$ for $n=1,2,\ldots,N$ with $N\to\infty$. The corresponding value is denoted by $\{\sigma\}(q)_\infty$ and may be taken, for example, from the work of Littlewood [7, p 124] or Macdonald [8, p 28].

It remains to rewrite the character $\{\sigma\}(\boldsymbol{x})_{2k}$ of GL($2k$) in terms of characters of $[\mu](\boldsymbol{x})_{2k+1}$ of SO($2k+1$). This is readily accomplished by using Schur function methods involving certain specific infinite series of Schur functions [15]. Under the appropriate restrictions of group elements we have

$$\mathrm{GL}(2k)\to\mathrm{SO}(2k) \qquad \{\sigma\}(\boldsymbol{x})_{2k}=[\sigma/D](\boldsymbol{x})_{2k} \tag{2.3}$$

$$\mathrm{SO}(2k)\to\mathrm{SO}(2k+1) \qquad [\tau](\boldsymbol{x})_{2k}=[\tau/L](\boldsymbol{x})_{2k+1} \tag{2.4}$$

giving the relationship

$$\mathrm{GL}(2k)\to\mathrm{SO}(2k+1) \qquad \{\sigma\}(\boldsymbol{x})_{2k}=[\sigma/H](\boldsymbol{x})_{2k+1} \tag{2.5}$$

where

$$H=DL=\sum_{\pi}(-1)^{|\pi|}\{\pi\} \tag{2.6}$$

in which the sum is taken over all partitions π and $|\pi|=\pi_1+\pi_2+\ldots$ is the weight of π.

Incorporating these results in (2.1) gives

$$\chi_{\mathrm{B}}(\boldsymbol{x}, q)=q^{-1}\sum_{\sigma}\{\sigma\}(q)_\infty[\sigma/H](\boldsymbol{x})_{2k+1}. \tag{2.7}$$

The coefficients appearing in Schur function products and quotients are identical [7, p 110; 8 p 39]:

$$\{\mu\}\{\pi\}=\sum_{\sigma} c^{\sigma}_{\mu\pi}\{\sigma\} \qquad \text{and} \qquad \{\sigma/\pi\}=\sum_{\mu} c^{\sigma}_{\mu\pi}\{\mu\}. \tag{2.8}$$

It then follows from (2.7) that

$$\chi_{\mathrm{B}}(\boldsymbol{x}, q)=q^{-1}\sum_{\mu}\{\mu\}(q)_\infty H(q)_\infty[\mu](\boldsymbol{x})_{2k+1}. \tag{2.9}$$

The principal specialisation referred to above gives [8, p 28]

$$\{\mu\}(q)_\infty = q^{n(\mu)} \prod_{(i,j)}^{F^\mu} (1-q^{h_{ij}})^{-1} \qquad \text{where} \qquad n(\mu) = \sum_i i\mu_i. \tag{2.10}$$

The product is over all boxes of the Young diagram F^μ, each specified by row and column labels i and j, whilst h_{ij} is the hook length of the box in the (i, j)th position:

$$h_{ij} = \mu_i + \mu'_j - i - j + 1 \tag{2.11}$$

where μ' is the partition conjugate to μ in the sense that, just as μ_i is the length of the ith row of F^μ, so μ'_j is the length of the jth column of F^μ. The notion of hook length is illustrated in figure 1.

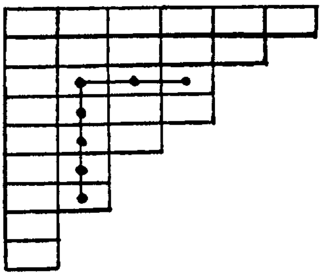

Figure 1. Illustration of the hook length formula (2.11) in the case $\mu = (654^2 32^2 1^2)$ and $(i, j) = (3, 2)$ for which $\mu_3 = 4$ and $\mu'_2 = 7$, leading to $h_{32} = 7$.

Finally the generating function for H [16] is such that

$$H(q)_\infty = \prod_{1\leq n<\infty} (1+q^n)^{-1} \prod_{1\leq m<n<\infty} (1-q^{m+n})^{-1} \tag{2.12}$$

so that

$$H(q)_\infty = 1 - q + q^4 + q^6 + 2q^8 + 3q^{10} + q^{11} + 6q^{12} + 2q^{13} + 9q^{14} + 6q^{15} + 16q^{16} + \dots. \tag{2.13}$$

It appears that the substitution of (2.10) and (2.12) into (2.9) would then yield the required expansion of the spectrum-generating function. However there is a subtle difference between (1.1) and (2.9). In (1.1) the summation is over all inequivalent irreducible characters χ^λ of SO$(D-1)$, whereas in (2.9) the summation is over all partitions μ. Unfortunately, the characters $[\mu](x)_{2k+1}$ of SO$(2k+1)$ specified by partitions μ are not all linearly independent. Indeed the complete set of standard inequivalent irreducible characters of SO$(2k+1)$ are those specified by partitions μ for which the number of parts μ'_1 satisfies the constraint $\mu'_1 \leq k$. If $\mu'_1 > k$ then recourse must be made to the modification rule [17]

$$[\mu](x)_{2k+1} = (-1)^{j-1}[\mu - h](x)_{2k+1} \qquad \text{with} \qquad h = 2\mu'_1 - 2k - 1 \tag{2.14}$$

where the Young diagram $F^{\mu-h}$ is obtained from F^μ by the deletion of a continuous boundary strip of boxes of length h starting at the foot of the first column and extending over j columns.

The application of this to (2.9) then gives

$$\chi_B(x, q) = q^{-1} \sum_{\substack{\lambda \\ \lambda'_1 \leq k}} \{\lambda^s_{2k+1}\}(q)_\infty H(q)_\infty [\lambda](x)_{2k+1} \tag{2.15}$$

where the signed sequence [18] can be written as

$$\{\lambda^s_{2k+1}\}=\sum_\mu (-1)^{s(\mu)}\{\mu\} \tag{2.16}$$

with a summation over all those μ such that the repeated application of (2.14) gives

$$[\mu](\boldsymbol{x})_{2k+1}=(-1)^{s(\mu)}[\lambda](\boldsymbol{x})_{2k+1} \tag{2.17}$$

where the sign factor $(-1)^{s(\mu)}=\pm 1$ depends upon μ. This formula is the required expansion of the spectrum-generating function for the bosonic string. It has been obtained using slightly different means by Curtright *et al* [3] who specified the signed sequence diagrammatically.

It follows from the modification rule (2.14) that

$$\{\lambda^s_{2k+1}\}=\sum_{m=0}^{\infty}\sum_{(j)}(-1)^{|j|-m}\{\lambda+h_{j_1}+h_{j_2}+\ldots+h_{j_m}\} \tag{2.18}$$

with

$$h_j=2k+1-2(\lambda'_j-j+1) \tag{2.19}$$

and

$$|j|=j_1+j_2+\ldots+j_m. \tag{2.20}$$

The second summation in (2.18) is over all possible sequences $(j)=(j_1,j_2,\ldots,j_m)$ such that $1\leqslant j_1<j_2<\ldots<j_m$ and the notation is such that $\lambda+h_{j_1}+h_{j_2}+\ldots+h_{j_m}$ specifies a Young diagram obtained from F^λ by the consecutive addition of continuous boundary strips of boxes of lengths h_j with $j=j_1,j_2,\ldots,j_m$, each starting at the foot of the jth column and extending back to the first column. It is not difficult to see, as illustrated in figure 2, that these additions taken in this order all overlap one another and do indeed always reach the first column, extending down as far as the p_jth row with $p_j=2k+1-(\lambda'_j-j+1)$. Although it is also possible to generalise the determinantal expansion

$$\{\lambda\}=|\{1^{\lambda'_j-j+i}\}| \tag{2.21}$$

to express the signed sequence in the form

$$\{\lambda^s_{2k+1}\}=|\{1^{\lambda'_j-j+i}\}+\{1^{2k-1-\lambda'_j+j+i}\}| \tag{2.22}$$

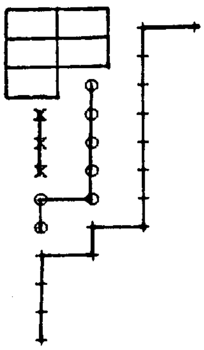

Figure 2. Illustration of the addition of consecutive boundary strips required in (2.18) in the case $\lambda=(2^21)$ and $k=4$, for which (2.19) gives $h_1=3$, $h_2=7$, $h_3=13$, $h_4=15,\ldots$. Typically taking $m=3$ and $j=(1,2,4)$ gives a contribution $(+1)$ $\{43^721^3\}$ corresponding to the diagram shown. Note that $p_1=6$, $p_2=8$ and $p_4=12$.

it does not seem possible to exploit this to obtain a succinct form of the principally specialised character required in (2.15). It appears that for arbitrary λ the optimum expression is (2.18). However, for the special case $\lambda=0$ we have

$$\begin{aligned}\{0^s_{2k+1}\} &= \{0\}+\{1^{2k+1}\}-\{21^{2k+1}\}-\{2^{2k+2}\}+\{31^{2k+2}\}\\ &\quad +\{32^{2k+1}1\}-\{3^2\,2^{2k+1}\}-\{3^{2k+3}\}-\ldots\\ &= \sum_{r=0}^{\infty}\sum_{(a)}(-1)^{|a|}\begin{pmatrix} a_1 & a_2 & \ldots & a_r\\ 2k+a_1 & 2k+a_2 & \ldots & 2k+a_r\end{pmatrix}\end{aligned} \tag{2.23}$$

where the Frobenius notation [7, p 60] for partitions has been used. The second summation is over all $\boldsymbol{a}=(a_1,a_2,\ldots,a_r)$ with $a_1>a_2>\ldots>a_r\geqslant 0$ and $|\boldsymbol{a}|=a_1+a_2+\ldots+a_r$.

In applying (2.15), as stressed elsewhere [2, 3], the signed sequence problem disappears in the limit as $k\to\infty$ since in this case no modifications are required. For finite k modifications must be taken into account in the case of sufficiently high levels. For given λ the lowest level, $L_{\mathrm{B}}(\lambda)$, at which $[\lambda](\boldsymbol{x})_{2k+1}$ makes an appearance in (2.15) is

$$L_{\mathrm{B}}(\lambda)=\sum_{i=1}^{\lambda_1'} i\lambda_i = n(\lambda). \tag{2.24}$$

Modified contributions first make their appearance at the level

$$L=L_{\mathrm{B}}(\lambda)+(k+1)(2k+1-2\lambda_1'). \tag{2.25}$$

For the SO$(2k+1)$ scalar character $[0](\boldsymbol{x})_{2k+1}$ this takes the value $(k+1)(2k+1)$, so that in the case $D=26$ and correspondingly $k=12$ the first modification to the coefficient of $[0](\boldsymbol{x})_{25}$ occurs at the level $L=325$. Unfortunately modifications are required at lower levels for other characters. The minimal case is provided by the SO$(2k+1)$ character $[1^k](\boldsymbol{x})_{2k+1}$ for which the first modification occurs with $L=(k+1)(k+2)/2$, i.e. $L=91$ if $k=12$ and $L=15$ if $k=4$.

Since (2.12) can be written in the form

$$H(q)_\infty=(1-q)\prod_{2\leqslant m\leqslant n\leqslant\infty}(1-q^{m+n})^{-1} \tag{2.26}$$

and every non-empty Young diagram contains at least one box of hook length $h_{ij}=1$, it follows that the coefficients in the expansion of $\{\mu\}(q)_\infty H(q)_\infty$ in powers of q are all positive except for the single coefficient (-1) of q in the expansion of $\{0\}(q)_\infty H(q)_\infty$. This exception corresponds to the fact that for $L=1$ the bosonic string states span the vector representation of SO$(2k)$ whose character, when expressed in terms of SO$(2k+1)$ characters, takes the form $[1](\boldsymbol{x})_{2k}=[1](\boldsymbol{x})_{2k+1}-[0](\boldsymbol{x})_{2k+1}$. This is not the character of a complete representation of SO$(2k+1)$, attesting to the fact that the corresponding $L=1$ level of the string is necessarily massless.

Since all other coefficients of powers of q in the expansion of $\{\mu\}(q)_\infty H(q)_\infty$ are positive this would appear to provide a very simple proof of the fact that beyond the massless $L=1$ level the spectrum is composed of basis states which do span complete representations of SO$(2k+1)$. However it is an unfortunate aspect of the modification rule (2.14) that it involves sign factors which show up in the signed sequence (2.16) and thus preclude the possibility of proving directly that in the expansion of (2.15) all terms are indeed positive for $L>1$. For such massive states this must be the case, of course, but we are unable to supply a combinatorial proof of the fact.

Table 1. Bosonic spectrum: $\chi_B(x, q)$ expressed in terms of SO$(2k+1)$ characters $[\lambda](x)_{2k+1}$ for $k \geq 3$. The dimensions and number of states are given in the cases $k=4$ and 12 corresponding to SO(9) and SO(25), respectively.

	q^{-1}	1	q	q^2	q^3	q^4	q^5	Dimension SO(9)	Dimension SO(25)
[0]	−1	1			1		1	1	1
[1]		1				1	1	9	25
[2]			1		1		2	44	324
[3]				1		1	1	156	2 900
[4]					1		1	450	20 150
[5]						1		1 122	115 830
[6]							1	2 508	573 300
$[1^2]$				1		1		36	300
[21]					1	1	1	231	5 175
[31]						1	1	910	52 026
[41]							1	2 772	385 020
$[1^3]$							1	84	2 300
$[2^2]$							1	495	32 175
Number of states at each level in the case $k=4$	1	8	44	192	726	2 464	7 704		
Number of states at each level in the case $k=12$	1	24	324	3 200	25 650	176 256	1073 720		

The expansion of the bosonic spectrum-generating function $\chi_B(x, q)$ up to and including terms in q^5 is given in table 1, confirming earlier results [2]. Up to this level no modification rules are required for $k \geq 3$. The dimension of the irreducible representation corresponding to each character $[\lambda](x)_{2k+1}$ appearing in the table is displayed in two cases of subsequent interest, namely $D=10$ and $D=26$. The total number of states at each level is also displayed.

3. The Neveu–Schwarz string

We now turn our attention to strings involving both bosonic and fermionic excitation operators. In the case of the open Neveu-Schwarz string [5] in D-dimensional spacetime with $D=2k+2$ the vacuum state is once again an SO$(2k)$ singlet state which is tachyonic but now with mass squared $-k/8$. Excited states are now generated by the action of bosonic operators a^i_{-n} and fermionic operators $b^i_{-(n-1/2)}$ with $i=1, 2, \ldots, 2k$ and $n=1, 2, \ldots, \infty$. For fixed n both the set of bosonic operators and the set of fermionic operators transform as the basis states of the defining vector $2k$-dimensional representation of SO$(2k)$. Just as the operator a^i_{-n} contributes n to the mass squared value so the operator $b^i_{-(n-1/2)}$ contributes $n-\frac{1}{2}$. It follows that the spectrum-generating function of the Neveu-Schwarz string is then given by

$$\chi_{NS}(x, q) = q^{-k/8} \prod_{n=1}^{\infty} \prod_{x=1}^{2k} (1-x_i q^n)^{-1}(1+x_i q^{n-1/2}). \tag{3.1}$$

The factor $(1-x_iq^n)$ is associated with the operator a^i_{-n} and the factor $(1+x_iq^{n-1/2})$ is associated with the operator $b^i_{-(n-1/2)}$. The distinction between bosonic and fermionic operators is accounted for by the relevant powers being -1 and $+1$, respectively, allowing for multiple excitations of individual bosonic states but only single excitations of individual fermionic states.

One new ingredient over and above (2.2) which allows us to proceed as before is the analogous formula [7, p 103; 8, p 35]

$$\prod_{n=1}^{N}\prod_{i=1}^{2k}(1+x_iz_n)=\sum_{\tau}\{\tau'\}(z)_N\{\tau\}(x)_{2k} \tag{3.2}$$

where the summation is carried out over all partitions τ.

In applying (3.2) to (3.1) the required specialisation of the character $\{\tau'\}(z)$ is obtained by setting $z=q'$ with $q'_n=q^{n-1/2}$ for $n=1,2,\ldots,N$ and then taking the limit $N\to\infty$. Denoting this character by $\{\tau'\}(q')_\infty$ it is easy to see that

$$\{\tau'\}(q')_\infty=q^{-|\tau|/2}\{\tau'\}(q)_\infty \qquad \text{where} \qquad |\tau|=|\tau'|=\sum_i\tau_i.$$

Making use of (2.8) then gives

$$\chi_{\mathrm{NS}}(x,q)=q^{-k/8}\sum_{\pi,\sigma,\tau}c^{\pi}_{\sigma\tau}\{\sigma\}(q)_\infty\{\tau'\}(q')_\infty\{\pi\}(x)_{2k}. \tag{3.3}$$

Now, however, it should be noted that there exists a supersymmetric generalisation [19] of the symmetric functions known as Schur or S functions, namely the supersymmetric functions which might conveniently be called SS functions. Just as the S function $\{\mu\}(y)_N$ is the character of an irreducible representation of $\mathrm{U}(N)$ so the SS function $\{\mu\}(y/z)_{M/N}$ is the supercharacter of an irreducible representation of $\mathrm{U}(M/N)$. This supercharacter can be defined in terms of characters of $\mathrm{U}(M)\times\mathrm{U}(N)$ by means of the formula

$$\{\pi\}(y/z)_{M/N}=\sum_{\sigma,\tau}c^{\pi}_{\sigma\tau}\{\sigma\}(y)_M\{\tau'\}(z)_N. \tag{3.4}$$

Carrying out the same manipulations as before using (2.5) then enables the spectrum-generating function to be written in the form

$$\chi_{\mathrm{NS}}(x,q)=q^{-k/8}\sum_{\pi}\{\pi\}(q/q')_\infty[\pi/H](x)_{2k+1} \tag{3.5}$$

where $\{\pi\}(q/q')_\infty$ is the limit as $M\to\infty$ and $N\to\infty$ of $\{\pi\}(y/z)_{M/N}$ with the specialisation $y=q$ and $z=q'$. Once again taking advantage of (2.5) then gives

$$\chi_{\mathrm{NS}}(x,q)=q^{-k/8}\sum_{\mu}\{\mu\}(q/q')_\infty H(q/q')_\infty[\mu](x)_{2k+1}. \tag{3.6}$$

Now the specialisation defined above gives [7, p 125]

$$\{\mu\}(q/q')_\infty=\prod_{(i,j)}^{F^\mu}(q^i+q^{j-1/2})(1-q^{h_{ij}})^{-1}. \tag{3.7}$$

Furthermore, the generating function for the SS function series H [20] can be used to establish the specialisation

$$\begin{aligned}H(q/q')_\infty&=\prod_{1\leqslant m<\infty}(1+q^m)^{-1}(1+q^{m-1/2})^{-1}\prod_{1\leqslant m<n<\infty}(1-q^{m+n})^{-1}(1-q^{m+n-1})^{-1}\\&\quad\times\prod_{1\leqslant m,n<\infty}(1+q^{m+n-1/2}) \qquad (3.8)\\&=1-q^{1/2}+q^2+q^{7/2}+2q^4+3q^{11/2}+5q^6+2q^{13/2}+2q^7+9q^{15/2}+13q^8\ldots. \qquad (3.9)\end{aligned}$$

Once again it is necessary to use the modification rule (2.14) to standardise the SO$(2k+1)$ characters $[\mu](x)_{2k+1}$ in cases for which $\mu'_1 > k$. This yields the analogue of (2.15), namely

$$\chi_{\mathrm{NS}}(x, q) = q^{-k/8} \sum_{\substack{\lambda \\ \lambda'_1 \leq k}} \{\lambda^s_{2k+1}\}(q/q')_\infty H(q/q')_\infty [\lambda](x)_{2k+1} \tag{3.10}$$

where the signed sequence is given as before by each of (2.16), (2.18) and (2.22). In the absence of modifications the lowest level, $L_{\mathrm{NS}}(\lambda)$, for which $[\lambda](x)_{2k+1}$ makes an appearance in (3.10) is

$$L_{\mathrm{NS}}(\lambda) = \sum_{i=1}^{r} \{i(\lambda_i + \lambda'_i - 2i + 1) - (\lambda'_i - i + 1)/2\} \tag{3.11}$$

where r is the Frobenius rank of the partition [7, p 60]. Modification rules make their presence felt at the level

$$L = L_{\mathrm{NS}}(\lambda) + (2k + 1 - 2\lambda'_1)/2. \tag{3.12}$$

Table 2. GSO-projected Neveu-Schwarz spectrum: $\frac{1}{2}\{\chi_{\mathrm{NS}}(x, q) - \chi'_{\mathrm{NS}}(x, q)\}$ expressed in terms of SO(9) characters $[\lambda](x)_9$.

	1	q	q^2	q^3	q^4	q^5	Dimension
[0]	−1			1	1	3	1
[1]	1		1	1	4	5	9
[2]		1		2	2	9	44
[3]			1		3	3	156
[4]				1		3	450
[5]					1		1 122
[6]						1	2 508
$[1^2]$			1	2	4	8	36
[21]			1	2	5	10	231
[31]				1	3	7	910
[41]					1	3	2 772
[51]						1	7 140
$[1^3]$		1		2	3	10	84
$[21^2]$			1	1	5	9	594
$[31^2]$				1	1	7	2 457
$[41^2]$					1	1	7 700
$[51^2]$						1	20 196
$[1^4]$			1	1	4	7	126
$[21^3]$				2	3	10	924
$[31^3]$					2	4	3 900
$[41^3]$						2	12 375
$[2^2]$				1	1	5	495
[32]					1	2	2 574
[42]						1	8 748
$[2^21]$†					2	4	1 650
[321]†						2	9 009
$[2^21^2]$†					1	4	2 772
$[321^2]$†						1	15 444
$[2^3]$†						1	1 980
Number of states at each level	8	128	1 152	7 680	42 112	200 448	

† These five entries complete the fourth and fifth levels of the tabulation of [2].

For the SO$(2k+1)$ scalar character $[0](\boldsymbol{x})_{2k+1}$ this takes the value $(2k+1)/2$ so that in the case $D=10$ the first modification to the coefficient of $[0](\boldsymbol{x})_9$ occurs at the level $L=\frac{9}{2}$. The minimal case is provided once again by the SO$(2k+1)$ character $[1^k](\boldsymbol{x})_{2k+1}$ for which the first modification occurs with $L=(k+1)/2$, i.e. $L=\frac{5}{2}$ if $k=4$.

Rather than tabulate the complete expansion of the Neveu-Schwarz spectrum-generating function, $\chi_{\rm NS}(x, q)$, given by (3.10), we give in table 2 just those terms associated with integer powers of q. The reason for this restriction is made clear in § 4. In drawing up table 2 the modifications (2.14) appropriate to SO(9) have been used in the calculation of the entries. Thus the resulting expansion only applies to the case $k=4$. However the expansion is complete in this case up to and including the sixth level, thus extending previous tabulations [2, 21].

4. The Ramond string

Turning now to the Ramond string [6] in D-dimensional spacetime with $D=2k+2$, the vacuum state is 2^k-dimensional and transforms as the basis of the reducible spin representation of SO$(2k)$, whose irreducible constituents are the two inequivalent 2^{k-1}-dimensional spin representations. This vacuum state is massless. Excited states are now generated by the action of bosonic operators a^i_{-n} and fermionic operators d^i_{-n} with $i=1,2,\ldots,2k$ and $n=1,2,\ldots,\infty$. For each fixed n both sets of bosonic and fermionic operators transform as the basis states of the $2k$-dimensional vector representation of SO$(2k)$. Their contributions to the mass squared value is n. It follows that the spectrum-generating function for the Ramond string then takes the form

$$\chi_{\rm R}(\boldsymbol{x};\, q)=\Delta(\boldsymbol{x})_{2k}\prod_{n=1}^{\infty}\prod_{i=1}^{2k}(1-x_iq^n)^{-1}(1+x_iq^n) \tag{4.1}$$

where

$$\Delta(\boldsymbol{x})_{2k}=\prod_{i=1}^{k}(x_i^{1/2}+x_i^{-1/2}) \tag{4.2}$$

is the character of the reducible 2^k-dimensional spin representation of SO$(2k)$. The terms in the product are associated with the bosonic and fermionic operators in an obvious way.

Making use of (2.2), (2.5) and (3.2) along with the specialisations defined earlier leads to the formula

$$\chi_{\rm R}(\boldsymbol{x}, q)=\sum_{\pi}\{\pi\}(q/q)_{\infty}\Delta(\boldsymbol{x})_{2k+1}[\pi/H](\boldsymbol{x})_{2k+1} \tag{4.3}$$

where it is to be noted that use has been made of the fact that the character (4.2) of the reducible spin representation of SO$(2k)$ coincides with that of the irreducible spin representation of SO$(2k+1)$. The product of the two characters of SO$(2k+1)$ appearing in (4.3) may be expressed in terms of irreducible characters by making use of the formula [15]

$$\Delta(\boldsymbol{x})_{2k+1}[\pi](\boldsymbol{x})_{2k+1}=[\Delta;\, \pi/Q](\boldsymbol{x})_{2k+1} \tag{4.4}$$

where Q is an infinite series of S functions such that

$$HQ=B=\sum_{\beta}\{\beta\} \tag{4.5}$$

with the summation to be carried out over those partitions β for which β_j' is even for all j. It then follows that

$$\chi_{\mathrm{R}}(\boldsymbol{x}, q) = \sum_{\mu} \{\mu\}(q/q)_\infty B(q/q)_\infty [\Delta; \mu](\boldsymbol{x})_{2k+1}. \tag{4.6}$$

The required specialisation then gives [7, p 125]

$$\{\mu\}(q/q)_\infty = \prod_{(ij)}^{F^\mu} (q^i + q^j)(1 - q^{h_{ij}})^{-1} \tag{4.7}$$

and [20]

$$B(q/q)_\infty = \prod_{1 \leqslant m < n < \infty} (1 + q^{m+n})(1 - q^{m+n})^{-1} \tag{4.8}$$

$$= 1 + 2q^2 + 4q^3 + 8q^4 + 16q^5 + 32q^6 + 60q^7 + 114q^8 + 212q^9 + 384q^{10} + 692q^{11} + \ldots. \tag{4.9}$$

Once again the summation in (4.6) is over all partitions μ and the corresponding characters $[\Delta; \mu](\boldsymbol{x})_{2k+1}$ are not all linearly independent. This time the modification rule appropriate to the case $\mu_1' > k$ takes the form [17]

$$[\Delta; \mu](\boldsymbol{x})_{2k+1} = (-1)^j [\Delta; \mu - h](\boldsymbol{x})_{2k+1} \qquad \text{with} \qquad h = 2\mu_1' - 2k - 2 \tag{4.10}$$

where the notation is the same as that of (2.14). The application of this to (4.6) then gives

$$\chi_{\mathrm{R}}(\boldsymbol{x}, q) = \sum_{\lambda} \{\lambda_{2k+1}^{\mathrm{sp}}\}(q/q)_\infty B(q/q)_\infty [\Delta; \lambda](\boldsymbol{x})_{2k+1} \tag{4.11}$$

where the signed sequence arising from the modification of spinor characters can be written as

$$\{\lambda_{2k+1}^{\mathrm{sp}}\} = \sum_{\mu} (-1)^{s(\mu)} \{\mu\} \tag{4.12}$$

with the summation over all those μ such that the repeated application of (4.10) gives

$$[\Delta; \mu](\boldsymbol{x})_{2k+1} = (-1)^{s(\mu)} [\Delta; \lambda](\boldsymbol{x})_{2k+1}. \tag{4.13}$$

The analogue of (2.18) is then

$$\{\lambda_{2k+1}^{\mathrm{sp}}\} = \sum_{m=0}^{\infty} \sum_{(\boldsymbol{j})} (-1)^{|\boldsymbol{j}|} \{\lambda + h_{j_1} + h_{j_2} + \ldots + h_{j_m}\} \tag{4.14}$$

with

$$h_j = 2k + 2 - 2(\lambda_j' - j + 1) \tag{4.15}$$

and

$$|\boldsymbol{j}| = j_1 + j_2 + \ldots + j_m \tag{4.16}$$

and that of (2.22) is

$$\{\lambda_{2k+1}^{\mathrm{sp}}\} = \left| \{1^{\lambda_j' - j + i}\} - \{1^{2k - \lambda_j' + j + i}\} \right|. \tag{4.17}$$

This time in the special case $\lambda = 0$ we have

$$\{0_{2k+1}^{\mathrm{sp}}\} = \{0\} - \{1^{2k+2}\} + \{21^{2k+2}\} - \{2^{2k+3}\} + \ldots$$

$$= \sum_{r=0}^{\infty} \sum_{(\boldsymbol{a})} (-1)^{|\boldsymbol{a}|+r} \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ 2k+1+a_1 & 2k+1+a_2 & \ldots & 2k+1+a_r \end{pmatrix}. \tag{4.18}$$

Table 3. GSO-projected Ramond spectrum: $\frac{1}{2}\{\chi_{\mathrm{R}}(x,q)+\chi'_{\mathrm{R}}(x,q)\}$ expressed in terms of SO(9) characters $[\Delta;\lambda](x)_9$.

	1	q	q^2	q^3	q^4	q^5	Dimension
$[\Delta;0]'$	$-\frac{1}{2}$						0
$[\Delta;0]$	$\frac{1}{2}$		1	2	4	8	16
$[\Delta;1]$		1	1	3	7	15	128
$[\Delta;2]$			1	2	5	12	576
$[\Delta;3]$				1	2	6	1 920
$[\Delta;4]$					1	2	5 280
$[\Delta;5]$						1	12 672
$[\Delta;1^2]$			1	2	5	12	432
$[\Delta;21]$				1	4	10	2 560
$[\Delta;31]$					1	4	9 504
$[\Delta;41]$						1	27 648
$[\Delta;1^3]$				1	2	6	768
$[\Delta;21^2]$					1	4	5 040
$[\Delta;31^2]$†						1	19 712
$[\Delta;1^4]$					1	2	672
$[\Delta;21^3]$						1	4 608
$[\Delta;2^2]$†						2	4 928
Number of states at each level	8	128	1 152	7 680	42 112	200 448	

† These two entries complete the fifth level in the tabulation of [2].

The lowest level, $L_{\mathrm{R}}(\Delta;\lambda)$, at which $[\Delta;\lambda](x)_{2k+1}$ makes an appearance in (4.11) is

$$L_{\mathrm{R}}(\Delta;\lambda)=\sum_{i=1}^{r} i(\lambda_i+\lambda'_i-2i+1) \tag{4.19}$$

where r is the Frobenius rank of λ. Modified contributions enter for the first time at the level

$$L=L_{\mathrm{R}}(\Delta;\lambda)+(2k+2-2\lambda'_1). \tag{4.20}$$

For the SO$(2k+1)$ spinor character $\Delta(x)_{2k+1}$ this takes the value $(2k+2)$ so that in the case of $D=10$ and, correspondingly, $k=4$ the first modification to the coefficient of $\Delta(x)_q$ occurs at the level $L=10$. The minimal case is provided by the SO$(2k+1)$ character $[\Delta;1^k](x)_{2k+1}$, for which the first modification occurs with $L=(k+2)$, i.e. $L=6$ for $k=4$.

Rather than tabulate the expansion of the complete Ramond spectrum-generating function, $\chi_{\mathrm{R}}(x,q)$, we give in table 3 the expansion corresponding to the case $k=4$ with all entries associated with excited states halved. Once again the reason for making this restriction will be made clear in § 5. The expansion covers the ground state and the first five excited states, thereby extending previous tabulations [2, 21]. Since the modification rule (4.10) appropriate to SO(9) has been used in calculating the entries, the results apply only to the case $k=4$.

5. The Green–Schwarz superstring

Although the Neveu–Schwarz string model and the Ramond string model described in §§ 4 and 5 each involve bosonic and fermionic excitation operators, neither of them is spacetime supersymmetric. A very significant advance was made by Gliozzi *et al* [9] who combined these two models but restricted the Neveu-Schwarz sector to states of even G-parity and the Ramond sector to one for which the ground state is a Majorana-Weyl spinor. They did this by applying what has become known as the GSO projection. For general spacetime dimension D the corresponding Neveu-Schwarz-Ramond spectrum-generating function takes the form

$$\chi_{\mathrm{NSR}}(\boldsymbol{x}, q) = \chi^{\mathrm{B}}_{\mathrm{NSR}}(\boldsymbol{x}, q) + \chi^{\mathrm{F}}_{\mathrm{NSR}}(\boldsymbol{x}, q) \tag{5.1}$$

where the generating function for the bosonic Neveu-Schwarz sector is given by

$$\chi^{\mathrm{B}}_{\mathrm{NSR}}(\boldsymbol{x}, q) = \tfrac{1}{2}\{\chi_{\mathrm{NS}}(\boldsymbol{x}, q) + (-1)^{k/4}\chi'_{\mathrm{NS}}(\boldsymbol{x}, q)\} \tag{5.2}$$

with

$$\chi_{\mathrm{NS}}(\boldsymbol{x}, q) = q^{-k/8} \prod_{n=1}^{\infty} \prod_{i=1}^{2k} (1 - x_i q^n)^{-1}(1 + x_i q^{n-1/2}) \tag{5.3}$$

as in (3.1), and

$$\chi'_{\mathrm{NS}}(\boldsymbol{x}, q) = -q^{k/8} \prod_{n=1}^{\infty} \prod_{i=1}^{2k} (1 - x_i q^n)^{-1}(1 - x_i q^{n-1/2}). \tag{5.4}$$

Similarly, that of the fermionic Ramond sector is given by

$$\chi^{\mathrm{F}}_{\mathrm{NSR}}(\boldsymbol{x}, q) = \tfrac{1}{2}\{\chi_{\mathrm{R}}(\boldsymbol{x}, q) + \chi'_{\mathrm{R}}(\boldsymbol{x}, q)\} \tag{5.5}$$

with

$$\chi_{\mathrm{R}}(\boldsymbol{x}, q) = \Delta(\boldsymbol{x})_{2k} \prod_{n=1}^{\infty} \prod_{i=1}^{2k} (1 - x_i q^n)^{-1}(1 + x_i q^n) \tag{5.6}$$

as in (4.1), and

$$\chi'_{\mathrm{R}}(\boldsymbol{x}, q) = \Delta'(\boldsymbol{x})_{2k} \tag{5.7}$$

where

$$\Delta(\boldsymbol{x})_{2k} = \prod_{i=1}^{k} (x_i^{1/2} + x_i^{-1/2}) \qquad \text{and} \qquad \Delta'(\boldsymbol{x})_{2k} = \prod_{i=1}^{k} (x_i^{1/2} - x_i^{-1/2}). \tag{5.8}$$

It was observed [9] that in the case $k = 4$ corresponding to $D = 10$ this model is both free of tachyons and apparently supersymmetric in the sense that the partition functions for the Neveu-Schwarz and Ramond sectors are identical, thanks to the '*aequatio identica satis abstrusa*' due to Jacobi [10, p 147]

$$\tfrac{1}{2}q^{-1/2}\left(\prod_{n=1}^{\infty} (1 + q^{n-1/2})^8 - \prod_{n=1}^{\infty} (1 - q^{n-1/2})^8\right) = 8 \prod_{n=1}^{\infty} (1 + q^n)^8. \tag{5.9}$$

This is where the matter stood until a manifestly supersymmetric version of this model was developed by Green and Schwarz [11]. At first sight their model looks quite different since both the ground state and the excitation operators are different from those associated with the Neveu-Schwarz and Ramond models. The model was constructed in the lightcone gauge in ten-dimensional spacetime. The relevant transverse symmetry group is SO(8).

There exist three inequivalent fundamental eight-dimensional irreducible representations of SO(8). These are the vector representation, 8_v, and two spinor representations, 8_s and 8_c, specified by the Dynkin labels (1000), (0010) and (0001), respectively. These representations are also denoted [22] by [1], Δ_+ and Δ_-. Their characters are given in terms of the eigenvalues x_i and x_i^{-1}, for $i=1,2,3$ and 4, of an arbitrary SO(8) group element, by

$$8_v \qquad [1](\boldsymbol{x})_8=\sum_{i=1}^{4}(x_i+x_i^{-1}) \tag{5.10a}$$

$$8_s \qquad \Delta_+(\boldsymbol{x})_8=\tfrac{1}{2}\left(\prod_{i=1}^{4}(x_i^{1/2}+x_i^{-1/2})+\prod_{i=1}^{4}(x_i^{1/2}-x_i^{-1/2})\right) \tag{5.10b}$$

$$8_c \qquad \Delta_-(\boldsymbol{x})_8=\tfrac{1}{2}\left(\prod_{i=1}^{4}(x_i^{1/2}+x_i^{-1/2})-\prod_{i=1}^{4}(x_i^{1/2}-x_i^{-1/2})\right). \tag{5.10c}$$

This triality of SO(8) is such that, if group elements are parametrised by $\boldsymbol{y}$ and $\boldsymbol{z}$ rather than $\boldsymbol{x}$, where

$$\begin{array}{ll} y_1=y_5^{-1}=x_1^{1/2}x_2^{1/2}x_3^{1/2}x_4^{1/2} & z_1=z_5^{-1}=x_1^{1/2}x_2^{1/2}x_3^{1/2}x_4^{-1/2} \\ y_2=y_6^{-1}=x_1^{1/2}x_2^{1/2}x_3^{-1/2}x_4^{-1/2} & z_2=z_6^{-1}=x_1^{1/2}x_2^{1/2}x_3^{-1/2}x_4^{1/2} \\ y_3=y_7^{-1}=x_1^{1/2}x_2^{-1/2}x_3^{1/2}x_4^{-1/2} & z_3=z_7^{-1}=x_1^{1/2}x_2^{-1/2}x_3^{1/2}x_4^{1/2} \\ y_4=y_8^{-1}=x_1^{1/2}x_2^{-1/2}x_3^{-1/2}x_4^{1/2} & z_4=z_8^{-1}=x_1^{1/2}x_2^{-1/2}x_3^{-1/2}x_4^{-1/2} \end{array} \tag{5.11}$$

then

$$8_v \qquad [1](\boldsymbol{x})_8=\Delta_+(\boldsymbol{y})_8=\Delta_+(\boldsymbol{z})_8 \tag{5.12a}$$

$$8_s \qquad \Delta_+(\boldsymbol{x})_8=[1](\boldsymbol{y})_8=\Delta_-(\boldsymbol{z})_8 \tag{5.12b}$$

$$8_c \qquad \Delta_-(\boldsymbol{x})_8=\Delta_-(\boldsymbol{y})_8=[1](\boldsymbol{z})_8. \tag{5.12c}$$

The open superstring model of Green and Schwarz involves a ground state transforming as the reducible SO(8) representation 8_v+8_s. Excited states are generated by the action of the bosonic operators a^i_{-n} and the fermionic operators s^a_{-n}. Once again $n=1,2,\ldots,\infty$, whilst both i and a range over the values $1,2,\ldots,8$. For fixed n, a^i_{-n} with $i=1,2,\ldots,8$ transform as the basis states of the vector representation 8_v and s^a_{-n} with $a=1,2,\ldots,8$ transform as the basis states of the spinor representation 8_c. Their contribution to the mass squared value is in each case given by n. The spectrum-generating function for the Green–Schwarz superstring is then given by

$$\chi_{\mathrm{GS}}(\boldsymbol{x},q)=([1](\boldsymbol{x})_8+\Delta_+(\boldsymbol{x})_8)\prod_{n=1}^{\infty}\prod_{i=1}^{8}(1-x_iq^n)^{-1}(1+z_iq^n) \tag{5.13}$$

where the components of $\boldsymbol{z}$ are given in terms of those of $\boldsymbol{x}$ by (5.11). This complication means that it is a non-trivial exercise to expand the spectrum-generating function (5.13) in the required form (1.1). However, it should be noted that the SO(8) triality relations (5.12) are such that (5.13) can be written in the form

$$\chi_{\mathrm{GS}}(\boldsymbol{x},q)=\Delta(\boldsymbol{z})_8\prod_{n=1}^{\infty}\prod_{i=1}^{8}(1-x_iq^n)^{-1}(1+z_iq^n). \tag{5.14}$$

In order to establish that the Green-Schwarz model has precisely the same spectrum as the $D=10$ GSO-projected Neveu-Schwarz-Ramond model it is convenient to introduce the function

$$\phi(q)=\prod_{n=1}^{\infty}(1-q^n) \tag{5.15}$$

and the theta functions [10, p 228; 23, p 469; 24, p 69]

$$\begin{aligned}
\mathrm{i}\theta_1(\xi,\tau)&=q^{1/8}(x^{1/2}-x^{-1/2})\prod_{n=1}^{\infty}(1-q^n)(1-xq^n)(1-x^{-1}q^n)\\
\theta_2(\xi,\tau)&=q^{1/8}(x^{1/2}+x^{-1/2})\prod_{n=1}^{\infty}(1-q^n)(1+xq^n)(1+x^{-1}q^n)\\
\theta_3(\xi,\tau)&=\prod_{n=1}^{\infty}(1-q^n)(1+xq^{n-1/2})(1+x^{-1}q^{n+1/2})\\
\theta_4(\xi,\tau)&=\prod_{n=1}^{\infty}(1-q^n)(1-xq^{n-1/2})(1-x^{-1}q^{n-1/2})
\end{aligned} \tag{5.16}$$

where

$$q=\exp(\mathrm{i}2\pi\tau) \qquad x=\exp(\mathrm{i}2\pi\xi). \tag{5.17}$$

Furthermore, defining theta function products by

$$\theta_j(\boldsymbol{\xi},\tau)=\prod_{i=1}^{k}\theta_j(\xi_i,\tau) \qquad \text{for} \qquad j=1,2,3,4 \tag{5.18}$$

and introducing

$$\psi(\boldsymbol{\xi},\tau)=\prod_{n=1}^{\infty}\prod_{i=1}^{k}(1-q^n)^{-1}(1-x_iq^n)^{-1}(1-x_i^{-1}q^n)^{-1} \tag{5.19}$$

it is possible to rewrite (2.1), (5.3)-(5.7) and (5.13) in terms of these functions. The resulting expressions take the form

$$\chi_{\mathrm{B}}(\boldsymbol{x},q)=q^{-1}\phi(q)\psi(\boldsymbol{\xi},\tau) \tag{5.20}$$

$$\chi_{\mathrm{NS}}(\boldsymbol{x},q)=q^{-k/8}\theta_3(\boldsymbol{\xi},\tau)\psi(\boldsymbol{\xi},\tau) \tag{5.21a}$$

$$\chi'_{\mathrm{NS}}(\boldsymbol{x},q)=q^{-k/8}\theta_4(\boldsymbol{\xi},\tau)\psi(\boldsymbol{\xi},\tau) \tag{5.21b}$$

$$\chi_{\mathrm{R}}(\boldsymbol{x},q)=q^{-k/8}\theta_2(\boldsymbol{\xi},\tau)\psi(\boldsymbol{\xi},\tau) \tag{5.21c}$$

$$\chi'_{\mathrm{R}}(\boldsymbol{x},q)=\mathrm{i}^kq^{-k/8}\theta_1(\boldsymbol{\xi},\tau)\psi(\boldsymbol{\xi},\tau) \tag{5.21d}$$

where now

$$x_j=\exp(\mathrm{i}2\pi\xi_j) \qquad \text{for} \qquad j=1,2,\ldots,k. \tag{5.22}$$

Turning now to the Green-Schwarz spectrum-generating function, (5.14) can be written in the form

$$\chi_{\mathrm{GS}}(\boldsymbol{x},q)=q^{-1/2}\theta_2(\boldsymbol{\zeta},\tau)\psi(\boldsymbol{\xi},\tau) \tag{5.23}$$

with

$$z_j=\exp(\mathrm{i}2\pi\zeta_j) \qquad \text{for} \qquad j=1,2,3,4 \tag{5.24}$$

and x and z are related by (5.11). Now the equivalence of the Green–Schwarz model and the $D=10$ GSO-projected Neveu–Schwarz–Ramond model follows immediately from the theta function identity due to Jacobi [10, p 505; 23, p 468]

$$\theta_2(\zeta, \tau)=\tfrac{1}{2}\{\theta_3(\xi, \tau)-\theta_4(\xi, \tau)+\theta_2(\xi, \tau)+\theta_1(\xi, \tau)\} \tag{5.25}$$

where use has been made of the eveness or oddness of the various theta functions. This implies, on comparing (5.21) in the case $D=10$ and correspondingly $k=4$ with (5.14), that

$$\begin{aligned}\chi_{\mathrm{GS}}(x, q)&=\chi_{\mathrm{NSR}}(x, q)\\ &=\tfrac{1}{2}\{\chi_{\mathrm{NS}}(x, q)-\chi'_{\mathrm{NS}}(x, q)+\chi_{\mathrm{R}}(x, q)+\chi'_{\mathrm{R}}(x, q)\}.\end{aligned} \tag{5.26}$$

This result seems to have first been obtained in this way by Nahm [12]. It is now clear that the required generalisation of Jacobi's abstruse identity (5.9) which would have considerably strengthened the arguments of Gliozzi *et al* [9] in favour of full spacetime supersymmetry is provided by the theta function identity (5.25)—again due to Jacobi. Of course, (5.9) follows from the special case of (5.25) obtained by setting $x_j=1$ or, equivalently, $\xi_j=0$ for $j=1, 2, 3, 4$.

A particular merit of the identity (5.26) is that it enables the spectrum-generating function for the Green–Schwarz superstring to be written in terms of SO(8) and SO(9) characters by means of the results of §§ 2–4. For $k=4$ the GSO projection (5.2) is such that in the Neveu–Schwarz sector the surviving terms in the expansion of (3.10) in the form (1.1) are those for which the mass squared value $L_0+L=L-\frac{1}{2}$ is an integer. Thus the terms proportional to half-odd-integer powers of q are to be discarded. This restriction is precisely the one used in § 3 in drawing up table 2, which thus gives the bosonic states of the lowest mass levels of the Green–Schwarz superstring. The tachyonic state is thus lost and the ground state is the massless state transforming as 8_v. In the same way the GSO projection (5.5) is such that in the Ramond sector the ground state is the massless state transforming as 8_s, whilst the excited states are obtained from (4.11) simply by halving the coefficients of all the characters of SO(9). This restriction coincides with that used in drawing up table 3, which thus gives the fermionic states of the lowest mass levels of the Green–Schwarz superstring.

The supersymmetry of the model shows itself in the equality between the number of bosonic and fermionic states at each level in tables 2 and 3.

6. Heterotic strings

All the string models discussed so far have been open string or open superstring models. Rather than consider the whole range of closed string models we now turn to heterotic strings [13]. These are closed orientable superstrings constructed as a chiral combination of $D=10$ right-moving GSO-projected Neveu–Schwarz–Ramond states together with $D=26$ left-moving bosonic string states compactified over a sixteen-dimensional torus in such a way as to leave $D=10$ left-moving bosonic string states carrying quantum numbers of a rank 16 gauge group G. The compactification involves sixteen-dimensional even self-dual lattices, Γ, of which there are just two [25, p 55], $\Gamma_8\times\Gamma_8$ and Γ_{16}, which are associated with the gauge groups $E_8\times E_8$ and Spin(32)/Z_2, respectively.

The spectrum-generating function for the right-moving sector is given by

$$\chi^{\mathrm{R}}_{\mathrm{HET}}(x, r)=\chi_{\mathrm{NSR}}(x, r)=\chi_{\mathrm{GS}}(x, r) \tag{6.1}$$

where r rather than q has been used as the variable whose exponent is the excitation level in this right-moving sector. This function has been fully discussed in § 5 and its expansion is given in tables 2 and 3.

The spectrum-generating function for the left-moving sector is more complicated. The ground state is a scalar. There exists a set of bosonic excitation operators a^i_{-n} with $i=1,2,\ldots,8$ and $n=1,2,\ldots,\infty$. As usual these operators, for each fixed n, transform as the basis states of the vector representation 8_v of SO(8). In addition there exist bosonic excitation operators $\tilde{a}^I_{-n}$ with $I=1,2,\ldots,16$ and $n=1,2,\ldots,\infty$. However these operators are scalars as far as SO(8) is concerned. They are each associated with the zero vector in the sixteen-dimensional even self-dual lattice. Finally there exist so-called momentum operators p^I, with $I=1,2,\ldots,16$, associated with the vectors specifying the lattice points of Γ.

In flat ten-dimensional spacetime the contribution to the value of the mass squared for both a^i_{-n} and $\tilde{a}^I_{-n}$ is given by n, and p^I contributes $(p^I)^2/2$. It follows that the left-moving spectrum-generating function can be written in the form

$$\chi^{\mathrm{L}}_{\mathrm{HET}}(\boldsymbol{x},\boldsymbol{t},q)=q^{-1}\prod_{n=1}^{\infty}\prod_{i=1}^{8}(1-x_iq^n)^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-16}\sum_{\boldsymbol{m}\in\Gamma}\boldsymbol{t}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2} \tag{6.2}$$

where q has been used as the parameter whose exponent gives the left-moving excitation level. The notation is such that

$$\boldsymbol{t}^{\boldsymbol{m}}=t_1^{m_1}t_2^{m_2}\ldots t_I^{m_I}\ldots \tag{6.3}$$

and the summation in (6.2) is carried out over all vectors $\boldsymbol{m}$ specifying points in the even self-dual lattice Γ. As usual, the components x_i of $\boldsymbol{x}$, with $i=1,2,3,4$, serve to parametrise the conjugacy classes of SO(8). The connection between the lattice Γ and the gauge group is such that each component t_I of $\boldsymbol{t}$ with $I=1,2,\ldots,16$ serves to parametrise the conjugacy classes of the relevant gauge group G. Thus in (6.2) $\boldsymbol{m}$ is to be interpreted as a weight vector of G.

The left-moving spectrum-generating function (6.2) can be written in the form

$$\chi^{\mathrm{L}}_{\mathrm{HET}}(\boldsymbol{x},\boldsymbol{t},q)=\chi_{\mathrm{B}}(\boldsymbol{x},q)\chi_{\mathrm{G}}(\boldsymbol{t},q) \tag{6.4}$$

where

$$\chi_{\mathrm{G}}(\boldsymbol{t},q)=\prod_{n=1}^{\infty}(1-q^n)^{-16}\sum_{\boldsymbol{m}\in\Gamma}\boldsymbol{t}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2}. \tag{6.5}$$

The lattice $\Gamma_8\times\Gamma_8$ coincides with the weight lattice of $\mathrm{E}_8\times\mathrm{E}_8$, whilst Γ_{16} coincides with the weight lattice of Spin(32)/Z_2. This is the weight lattice associated with the tensor representations $[\lambda]$ of SO(32), with $|\lambda|$ even, and the spinor representations $[\Delta;\lambda]_+$ and $[\Delta;\lambda]_-$, with $|\lambda|$ even and odd, respectively. Thus (6.5) may be evaluated by making direct use of these weight lattices [1]. However, an alternative procedure is provided by yet another set of theta function expansions [10, p 501; 23, p 464; 24; pp 60-3]:

$$\mathrm{i}\theta_1(\xi,\tau)=\sum_{m\in\mathbb{Z}}(-1)^m x^{m+1/2}q^{(m+1/2)^2/2} \tag{6.6a}$$

$$\theta_2(\xi,\tau)=\sum_{m\in\mathbb{Z}}x^{m+1/2}q^{(m+1/2)^2/2} \tag{6.6b}$$

$$\theta_3(\xi,\tau)=\sum_{m\in\mathbb{Z}}x^m q^{m^2/2} \tag{6.6c}$$

$$\theta_4(\xi,\tau)=\sum_{m\in\mathbb{Z}}(-1)^m x^m q^{m^2/2}. \tag{6.6d}$$

It follows that

$$i^k\theta_1(\boldsymbol{\xi}, \tau) = \sum_{\boldsymbol{m}\in(\mathbb{Z}-1/2)^k} (-1)^{|\boldsymbol{m}|+k/2}\boldsymbol{x}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2} \tag{6.7a}$$

$$\theta_2(\boldsymbol{\xi}, \tau) = \sum_{\boldsymbol{m}\in(\mathbb{Z}-1/2)^k} \boldsymbol{x}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2} \tag{6.7b}$$

$$\theta_3(\boldsymbol{\xi}, \tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \boldsymbol{x}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2} \tag{6.7c}$$

$$\theta_4(\boldsymbol{\xi}, \tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} (-1)^{|\boldsymbol{m}|}\boldsymbol{x}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2} \tag{6.7d}$$

where $|\boldsymbol{m}| = m_1+m_2+\ldots+m_k$.

For $k=0 \pmod 8$ the lattice Γ_k is even and is defined [25, p 51] by

$$\Gamma_k = \{\boldsymbol{m}\,|\,2m_i\in\mathbb{Z},\ m_i-m_j\in\mathbb{Z},\ |\boldsymbol{m}|/2\in\mathbb{Z},\ i,j=1,2,\ldots,k\}. \tag{6.8}$$

It follows from (6.7) and (6.8) that for $k=0 \pmod 8$

$$\sum_{\boldsymbol{m}\in\Gamma_k} \boldsymbol{x}^{\boldsymbol{m}}q^{(\boldsymbol{m}\cdot\boldsymbol{m})/2} = \tfrac{1}{2}\{\theta_1(\boldsymbol{\xi},\tau)+\theta_2(\boldsymbol{\xi},\tau)+\theta_3(\boldsymbol{\xi},\tau)+\theta_4(\boldsymbol{\xi},\tau)\}. \tag{6.9}$$

In applying this identity to (6.5) in the case $\Gamma=\Gamma_8\times\Gamma_8$ it is convenient to write $\boldsymbol{t}=(\boldsymbol{u},\boldsymbol{v})$ and then to extend $\boldsymbol{u}$ and $\boldsymbol{v}$ from eight-component vectors to sixteen-component vectors in such a way that for $\boldsymbol{u}$, for example,

$$u_{I+8} = u_I^{-1} = \exp(-i2\pi\alpha_I) \qquad \text{for} \qquad I=1,2,\ldots,8. \tag{6.10}$$

Then

$$\chi_{E_8\times E_8}(\boldsymbol{u},\boldsymbol{v},q) = \chi_{E_8}(\boldsymbol{u},q)\chi_{E_8}(\boldsymbol{v},q) \tag{6.11}$$

with the spectrum-generating function $\chi_{E_8}(\boldsymbol{u},q)$ given by

$$\chi_{E_8}(\boldsymbol{u},q) = \phi(q)^{-8}\,\tfrac{1}{2}\{\theta_1(\boldsymbol{\alpha},\tau)+\theta_2(\boldsymbol{\alpha},\tau)+\theta_3(\boldsymbol{\alpha},\tau)+\theta_4(\boldsymbol{\alpha},\tau)\}. \tag{6.12}$$

The expression (6.12) may be written in terms of characters of SO(16) by noting that from (5.16)

$$\begin{aligned}\chi_{E_8}(\boldsymbol{u},q) = \tfrac{1}{2}\Bigg(&\prod_{n=1}^{\infty}\prod_{I=1}^{16}(1+u_Iq^{n-1/2}) + \prod_{n=1}^{\infty}\prod_{I=1}^{16}(1-u_Iq^{n-1/2})\\ &+q\Delta(\boldsymbol{u})_{16}\prod_{n=1}^{\infty}\prod_{I=1}^{16}(1+u_Iq^{n}) + q\Delta'(\boldsymbol{u})_{16}\prod_{n=1}^{\infty}\prod_{I=1}^{16}(1-u_Iq^{n})\Bigg).\end{aligned} \tag{6.13}$$

Making use of (3.2), (2.3) and (4.4) it follows that

$$\chi_{E_8}(\boldsymbol{u},q) = \sum_{\substack{\mu\\|\mu|\text{even}}} \{\mu'\}(q')_\infty D(q)_\infty[\mu](\boldsymbol{u})_{16} + q\sum_{\mu}\{\mu'\}(q)_\infty F(q)_\infty[\Delta;\mu]_{(-1)^{|\mu|}}(\boldsymbol{u})_{16} \tag{6.14}$$

where [15]

$$D=\sum_{\delta}\{\delta\} \qquad DQ=F=\sum_{\zeta}\{\zeta\} \tag{6.15}$$

with the first summation over those partitions δ, all of whose parts δ_j are even, and the second over all partitions ζ. The generating functions for the series D and F [16] are such that

$$\begin{aligned}D(q)_\infty = 1&+q^2+q^3+3q^4+3q^5+7q^6+8q^7+16q^8+20q^9+35q^{10}\\ &+16q^{11}+77q^{12}+102q^{13}+161q^{14}+\ldots\end{aligned}$$

and

$$F(q)_\infty = 1+q+2q^2+4q^3+7q^4+12q^5+21q^6+34q^7+56q^8 +90q^9+143q^{10}+223q^{11}+348q^{12}+532q^{13}+\ldots.$$

For completeness we note that in general it is necessary to apply the following modification rules [17] analogous to those of (2.14) and (4.10):

$$[\mu](\boldsymbol{x})_{2k} = (-1)^{j-1}[\mu - h](\boldsymbol{x})_{2k} \qquad \text{with} \qquad h = 2\mu_1' - 2k \tag{6.16}$$

and

$$[\Delta;\mu]_\pm(\boldsymbol{x}) = (-1)^j[\Delta;\mu - h]_\mp(\boldsymbol{x})_{2k} \qquad \text{with} \qquad h = 2\mu_1' - 2k - 1. \tag{6.17}$$

Their application leads to

$$\begin{aligned}\chi_{\mathrm{E}_8}(\boldsymbol{u}, q) = &\sum_{\substack{\lambda \\ |\lambda|\text{even},\lambda_1'\leqslant 8}} \{\lambda_{16}^{s\prime}\}(q')_\infty D(q)_\infty[\lambda](\boldsymbol{u})_{16} \\ &+ q\sum_{\substack{\lambda \\ \lambda_1'\leqslant 8}} \{\lambda_{16}^{sp\prime}\}(q)_\infty F(q)_\infty[\Delta;\lambda]_{(-1)^{|\lambda|}}(\boldsymbol{u})_{16}\end{aligned} \tag{6.18}$$

where now the signed sequences are conjugated. The expansion of this spectrum-generating function in terms of characters of SO(16) is given in table 4. The first seven levels are displayed.

Finally, it is necessary to express the characters of SO(16) in terms of those of E_8. No simple formula exists for this step. However, there are several tabulations [26, 27] of the branching rules of representations of E_8 restricted to the subgroup SO(16). These lead to the results given in table 5 in which each irreducible representation of E_8 has been labelled by (λ) or $(\Delta;\lambda)$ where λ or $\Delta;\lambda$ is the highest weight of the set of irreducible representations contained in the restriction of the E_8 representation to SO(16) [26]. The table extends as far as the level $L=6$. As dictated by (6.11) the product of two such expansions yields the expansion of the complete $\mathrm{E}_8\times\mathrm{E}_8$ spectrum-generating function. This is expressed in terms of characters of $\mathrm{E}_8\times\mathrm{E}_8$ in table 6, which extends to the level $L=6$, thereby extending a previous tabulation [28] as far as the level $L=4$.

Turning now to the case of $\Gamma=\Gamma_{16}$, with the notation

$$t_{I+16} = t_I^{-1} = \exp(-\mathrm{i}2\pi\gamma_I) \qquad \text{for} \qquad I = 1, 2, \ldots, 16 \tag{6.19}$$

the application of (6.9) to (6.5) yields the formula

$$\chi_{D_{16}}(\boldsymbol{t}, q) = \phi(q)^{-16}\tfrac{1}{2}\{\theta_1(\gamma,\tau)+\theta_2(\gamma,\tau)+\theta_3(\gamma,\tau)+\theta_4(\gamma,\tau)\} \tag{6.20}$$

and hence

$$\begin{aligned}\chi_{D_{16}}(\boldsymbol{u}, q) = \tfrac{1}{2}\Bigg(&\prod_{n=1}^{\infty}\prod_{I=1}^{32}(1+u_I q^{n-1/2}) + \prod_{n=1}^{\infty}\prod_{I=1}^{32}(1-u_I q^{u-1/2}) \\ &+ q^2\Delta(\boldsymbol{u})_{32}\prod_{n=1}^{\infty}\prod_{I=1}^{32}(1+u_I q^n) + q^2\Delta'(\boldsymbol{u})_{32}\prod_{n=1}^{\infty}\prod_{I=1}^{32}(1-u_I q^n)\Bigg).\end{aligned} \tag{6.21}$$

This leads inexorably to

$$\begin{aligned}\chi_{D_{16}}(\boldsymbol{u}, q) = &\sum_{\substack{\mu \\ |\mu|\text{even}}} \{\mu'\}(q')_\infty D(q)_\infty[\mu](\boldsymbol{u})_{32} \\ &+ q^2\sum_{\mu}\{\mu'\}(q)_\infty F(q)_\infty[\Delta;\mu]_{(-1)^{|\mu|}}(\boldsymbol{u})_{32}\end{aligned} \tag{6.22}$$

Table 4. $\chi_{E_8}(x, q)$ expressed in terms of SO(16) characters $[\lambda](x)_{16}$ and $[\Delta; \lambda](x)_{16}$.

	1	q	q^2	q^3	q^4	q^5	q^6	Dimension
$[0]$	1		1	1	3	3	7	1
$[1^2]$		1	1	3	4	9	13	120
$[1^4]$			1	1	3	5	12	1 820
$[1^6]$				1	1	4	6	8 008
$[1^8]_+$					1	1	3	6 435
$[1^8]_-$					1	1	3	6 435
$[2]$			1	1	3	4	9	135
$[21^2]$				1	2	5	9	7 020
$[21^4]$					1	2	5	60 060
$[21^6]$						1	3	162 162
$[2^2]$					1	1	4	5 304
$[2^21^2]$						1	2	141 372
$[2^21^4]$							1	716 040
$[2^3]$							1	89 760
$[31]$						1	2	8 925
$[31^3]$							1	176 800
Number of tensor states at each level	1	120	2 076	17 344	106 630	528 608	2265 656	
$[\Delta; 0]_+$		1	1	2	4	7	12	128
$[\Delta; 1]_-$			1	2	4	8	15	1 920
$[\Delta; 1^2]_+$				1	2	5	10	13 312
$[\Delta; 1^3]_-$					1	2	5	56 320
$[\Delta; 1^4]_+$						1	2	161 280
$[\Delta; 1^5]_-$							1	326 144
$[\Delta; 2]_+$					1	2	5	15 360
$[\Delta; 21]_-$						1	3	141 440
$[\Delta; 21^2]_+$							1	670 208
Number of spinor states at each level		128	2 048	17 408	106 496	528 896	2265 088	
Total number of states at each level	1	248	4 124	34 752	213 126	1057 504	4530 744	

Table 5. E_8 spectrum: $\chi_{E_8}(x, q)$ expressed in terms of E_8 characters $(\lambda)(x)$ and $(\Delta; \lambda)(x)$.

	1	q	q^2	q^3	q^4	q^5	q^6	Dimension
(0)	1		1	1	2	2	4	1
(1^2)		1	1	2	3	5	7	248
(2)			1	1	2	3	6	3 875
(21^2)				1	1	3	4	30 380
(2^2)					1	1	3	27 000
$(\Delta; 2)_+$					1	1	2	147 250
(31)						1	2	779 247
(31^3)							1	2 450 240
Number of states at each level	1	248	4124	34 752	213 126	1057 504	4530 744	

Table 6. $E_8 \times E_8$ spectrum: $\chi_{E_8 \times E_8}(x, q)$ expressed in terms of $E_8 \times E_8$ characters specified by their dimensionality.

	1	q	q^2	q^3	q^4	q^5	q^6
1×1	1		2	2	5	6	13
$248 \times 1 + 1 \times 248$		1	1	3	5	10	16
$3875 \times 1 + 1 \times 3875$			1	1	3	5	11
248×248			1	2	5	10	20
$30\,380 \times 1 + 1 \times 30\,380$				1	1	4	6
$3875 \times 248 + 248 \times 3875$				1	2	5	10
$27\,000 \times 1 + 1 \times 27\,000$					1	1	4
$147\,250 \times 1 + 1 \times 147\,250$					1	1	3
$30\,380 \times 248 + 248 \times 30\,380$					1	2	6
3875×3875					1	2	5
$779\,247 \times 1 + 1 \times 779\,247$						1	2
$27\,000 \times 248 + 248 \times 27\,000$						1	2
$147\,250 \times 248 + 248 \times 147\,250$						1	2
$30\,380 \times 3875 + 3875 \times 30\,380$						1	2
$2450\,240 \times 1 + 1 \times 2450\,240$							1
$779\,247 \times 248 + 248 \times 779\,247$							1
$27\,000 \times 3875 + 3875 \times 27\,000$							1
$147\,250 \times 3875 + 3875 \times 147\,250$							1
$30\,380 \times 30\,380$							1
Total dimension at each level	1	496	69 752	2115 008	34 670 620	394 460 000	3499 148 222

and hence, via (6.16) and (6.17), to

$$\chi_{D_{16}}(\boldsymbol{u}, q) = \sum_{\substack{\lambda \\ |\lambda| \text{even}, \lambda_1' \leq 16}} \{\lambda_{16}^{s\prime}\}(q')_\infty D(q)_\infty[\lambda](\boldsymbol{u})_{32} + q^2 \sum_{\substack{\lambda \\ \lambda_1' \leq 16}} \{\lambda_{16}^{sp\prime}\}(q)_\infty F(q)_\infty[\Delta; \lambda]_{(-1)^{|\lambda|}}(\boldsymbol{u})_{32}. \tag{6.23}$$

This expansion in terms of characters of SO(32) or, more properly, of Spin(32)/Z_2, is developed as far as the level $L=6$ in table 7.

A check on these calculations is provided by the equality between the total number of states at any given level as given by the $E_8 \times E_8$ model and by the Spin(32)/Z_2 model. The results of tables 6 and 7 confirm this equality, which is a reflection of the fact that the partition functions of these two models are identical. This can be viewed as yet another consequence of Jacobi's abstruse identity (5.9). In fact, successive powers of (5.9) lead via the trivial identity

$$\prod_{n=1}^{\infty} (1+q^{n-1/2})(1-q^{n-1/2}) = \prod_{n=1}^{\infty} (1-q^{2n-1}) = \prod_{n=1}^{\infty} (1+q^n)^{-1} \tag{6.24}$$

to

$$\prod_{n=1}^{\infty} (1+q^{n-1/2})^{16} + \prod_{n=1}^{\infty} (1-q^{n-1/2})^{16} = 2^8 q \prod_{n=1}^{\infty} (1+q^n)^{16} + 2 \prod_{n=1}^{\infty} (1+q^n)^{-8} \tag{6.25a}$$

$$\prod_{n=1}^{\infty} (1+q^{n-1/2})^{24} - \prod_{n=1}^{\infty} (1-q^{n-1/2})^{24} = 2^{12} q^{3/2} \prod_{n=1}^{\infty} (1+q^n)^{24} + 48 q^{1/2} \tag{6.25b}$$

$$\prod_{n=1}^{\infty} (1+q^{n-1/2})^{32} + \prod_{n=1}^{\infty} (1-q^{n-1/2})^{32} = 2^{16} q^2 \prod_{n=1}^{\infty} (1+q^n)^{32} + 2^{10} q \prod_{n=1}^{\infty} (1+q^n)^{8} + 2 \prod_{n=1}^{\infty} (1+q^n)^{-16}. \tag{6.25c}$$

The partition function $P_{E_8}(q)$ for the E_8 spectrum is obtained from (6.13) by setting $u_I = 1$ for $I = 1, 2, \ldots, 16$. It follows from (6.25a) that

$$P_{E_8}(q) = 2^8 q \prod_{n=1}^{\infty} (1+q^n)^{16} + \prod_{n=1}^{\infty} (1+q^n)^{-8}. \tag{6.26}$$

The partition function $P_{E_8 \times E_8}(q)$ for the $E_8 \times E_8$ spectrum is then obtained from (6.11) by setting both u_I and $v_I = 1$ for $I = 1, 2, \ldots, 16$, so that

$$P_{E_8 \times E_8}(q) = P_{E_8}(q) \times P_{E_8}(q). \tag{6.27}$$

The Spin(32)/Z_2 spectrum partition function $P_{D_{16}}(q)$ is obtained in the same way from (6.21) by setting $u_I = 1$ for $I = 1, 2, \ldots, 32$, and it follows from (6.25c) that

$$P_{D_{16}}(q) = 2^{16} q^2 \prod_{n=1}^{\infty} (1+q^n)^{32} + 2^9 q \prod_{n=1}^{\infty} (1+q^n)^{8} + \prod_{n=1}^{\infty} (1+q^n)^{-16} \tag{6.28}$$

Table 7. Spin(32)/Z_2 spectrum: $\chi_{D_{16}}(\boldsymbol{x}, q)$ expressed in terms of SO(32) characters $[\lambda](\boldsymbol{x})_{32}$ and $[\Delta; \lambda](\boldsymbol{x})_{32}$.

	1	q	q^2	q^3	q^4	q^5	q^6	Dimension
$[0]$	1		1	1	3	3	7	1
$[1^2]$		1	1	3	4		13	496
$[1^4]$			1	1	3	5	11	35 960
$[1^6]$				1	1	3	5	906 912
$[1^8]$					1	1	3	10 518 300
$[1^{10}]$						1	1	64 512 240
$[1^{12}]$							1	225 792 840
$[2]$			1	1	3	4	9	527
$[21^2]$				1	2	5	9	122 264
$[21^4]$					1	2	5	5 501 880
$[21^6]$						1	2	96 282 900
$[21^8]$							1	822 531 060
$[2^2]$					1	1	4	86 768
$[2^21^2]$						1	2	11 269 368
$[2^21^4]$							1	336 226 000
$[2^3]$							1	6 678 144
$[31]$						1	2	138 105
$[31^3]$							1	13 290 816
Number of tensor states at each level	1	496	36 984	1 066 432	17 369 116	197 327 712	1749 861 312	
$[\Delta; 0]_+$			1	1	2	4	7	32 768
$[\Delta; 1]_-$				1	2	4	8	1 015 808
$[\Delta; 1^2]_+$					1	2	5	15 204 352
$[\Delta; 1^3]_-$						1	2	146 276 352
$[\Delta; 1^4]_+$							1	1015 808 000
$[\Delta; 2]_+$						1	2	16 252 928
$[\Delta; 21]_-$							1	324 042 752
Number of spinor states at each level			32 768	1048 576	17 301 504	197 132 288	1749 286 912	
Total number of states at each level	1	496	69 752	2 115 008	34 670 620	394 460 000	3499 148 224	

Table 8. Heterotic string spectra: $\chi_{\mathrm{HET}}(x, t, q, r) = \chi^{\mathrm{R}}_{\mathrm{HET}}(x, r)\ \chi^{\mathrm{L}}_{\mathrm{HET}}(x, t, q)$ expressed in terms of contributions from the various right-hand levels of $\chi^{\mathrm{R}}_{\mathrm{HET}}(x, r) = \chi_{\mathrm{GS}}(x, r)$ matched with the left-hand levels of $\chi^{\mathrm{L}}_{\mathrm{HET}}(x, t, q) = \chi_{\mathrm{B}}(x, q)\chi_{\mathrm{G}}(t, q)$. Contributions, signified by their dimensionalities, are taken for the right-hand sector from tables 2 and 3, and for the left-hand sector from table 1 and either table 6 or 7 as appropriate for $\mathrm{E}_8 \times \mathrm{E}_8$ or $\mathrm{Spin}(32)/Z_2$, respectively.

$\chi^{\mathrm{R}}_{\mathrm{HET}}(x, r)\chi^{\mathrm{L}}_{\mathrm{HET}}(x, t, q)$
$(8+8)(1\times 496+8\times 1)$
$(128+128)(1\times 69\,752+8\times 496+44\times 1)$
$(1152+1152)(1\times 2115\,008+8\times 69\,752+44\times 496+192\times 1)$
$(7680+7680)(1\times 34\,670\,620+8\times 2115\,008+44\times 69\,752+192\times 496+726\times 1)$
$(42\,112+42\,112)(1\times 394\,460\,000+8\times 34\,670\,620+44\times 2115\,008+192\times 69\,752+726\times 496+2464\times 1)$
$(200\,448+200\,448)(1\times 3499\,148\,224+8\times 394\,460\,000+44\times 34\,670\,620+192\times 2115\,008$ $+726\times 69\,752+2464\times 496+7704\times 1)$

which is clearly the square of (6.26). Thus, as claimed, the partition functions of $\mathrm{E}_8 \times \mathrm{E}_8$ and $\mathrm{Spin}(32)/Z_2$ coincide.

In the case of both heterotic string models described here it is essential to match up the excitation levels of the right-moving and left-moving sectors, as pointed out for all closed string models in the introduction. The required spectrum-generating functions take the form

$$\chi^{\mathrm{E}_8\times\mathrm{E}_8}_{\mathrm{HET}}(x, u, v, r, q) = \chi^{\mathrm{R}}_{\mathrm{HET}}(x, r)\chi^{\mathrm{L}}_{\mathrm{HET}}(x, u, v, q) \tag{6.29}$$

$$= \chi_{\mathrm{GS}}(x, r)\chi_{\mathrm{B}}(x, q)\chi_{\mathrm{E}_8}(u, q)\chi_{\mathrm{E}_8}(v, q) \tag{6.30}$$

and

$$\chi^{\mathrm{Spin}(32)/Z_2}_{\mathrm{HET}}(x, t, r, q) = \chi^{\mathrm{R}}_{\mathrm{HET}}(x, r)\chi^{\mathrm{L}}_{\mathrm{HET}}(x, t, q) \tag{6.31}$$

$$= \chi_{\mathrm{GS}}(x, r)\chi_{\mathrm{B}}(x, q)\chi_{D_{16}}(t, q). \tag{6.32}$$

In (6.30) and (6.32) it is only necessary to retain those terms for which the exponents of r and of q are equal. Amongst other things this has the effect of eliminating tachyon states. The content of the massless ground state and the first five excited states is given in table 8 where the first factor arises from $\chi^{\mathrm{R}}_{\mathrm{HET}}(x, r)$ and the second from $\chi^{\mathrm{L}}_{\mathrm{HET}}(x, u, v, q)$ or $\chi^{\mathrm{L}}_{\mathrm{HET}}(x, t, q)$. All contributions have been signified by means of their dimension. The two terms in the first factor are to be taken from tables 2 and 3. Each term in the second factor is a product of terms to be taken from table 1 in the case SO(9) and terms to be taken from table 6 or 7 as appropriate for the $\mathrm{E}_8 \times \mathrm{E}_8$ or $\mathrm{Spin}(32)/Z_2$ models. The final step is that of combining the SO(9) characters from the left and right sectors and expressing them linearly in terms of SO(9) characters. This step is straightforward enough once it is realised that in the second factor of each expression in table 8 the SO(9) character signified by 8 should be treated as $9-1$ and that each excited state contains its predecessor [1].

7. Conclusions

The spectrum-generating functions for strings and superstrings calculated in this paper show a remarkable similarity. This is due to the underlying presence in the analysis of various products and sums of theta functions. This is no accident, of course, since it is a necessary requirement, as stressed for example by Nahm [12], for the string models to exhibit modular invariance. Here we have not been directly concerned with modular invariance, nor have we been concerned with algebraic structures other than finite-dimensional simple Lie algebras, the characters of whose irreducible representations have served as the basis for all our expansions of spectrum-generating functions. Indeed we have deliberately eschewed the use of Kac–Moody algebras, although our results incorporate useful explicit information on the characters of some irreducible representations of these infinite-dimensional Lie algebras.

Apart from giving results appropriate to particular string and superstring models the merit of our approach involving theta functions and characters of $SO(2k)$ and $SO(2k+1)$ is that it is amenable to calculations for all values of k if modification rules are taken into account. For example, table 7 has been drawn up in the case $k=16$ appropriate to SO(32). However, for all the levels tabulated no modifications are required for this value of k. For larger values of k table 7 applies as it stands. For smaller values modification rules may be needed. For example, the results appropriate to SO(16) given in table 4 may be recovered from table 7 merely by noting that the modification rules of SO(16) given by (6.16) and (6.17) imply

$$[21^8]=[21^6] \qquad [1^{12}]=[1^4] \qquad [1^{10}]=[1^6] \tag{7.1}$$

whilst

$$[1^8]=[1^8]_+ + [1^8]_-. \tag{7.2}$$

These SO(16) results of table 4 are of relevance to $D=18$ models [14]. Equivalent SO(8) results may be obtained in the same way.

The k dependence of the GSO projection in the Neveu–Schwarz sector as defined by (5.2) is crucial in yet another model, namely the $D=26$ giant superstring model introduced by Thierry-Mieg [14]. The relevant transverse symmetry group is SO(24) and the spectrum-generating function takes the form

$$\chi_{\mathrm{TM}}(x, q, r)=\chi^{\mathrm{R}}_{\mathrm{TM}}(x, r)\chi^{\mathrm{L}}_{\mathrm{TM}}(x, q) \tag{7.3}$$

with

$$\chi^{\mathrm{R}}_{\mathrm{TM}}(x, r)=\chi_{\mathrm{B}}(x, r) \tag{7.4}$$

and

$$\chi^{\mathrm{L}}_{\mathrm{TM}}(x, q)=\tfrac{1}{2}\{\chi_{\mathrm{NS}}(x, q)-\chi'_{\mathrm{NS}}(x, q)+\chi_{\mathrm{R}}(x, q)-\chi'_{\mathrm{R}}(x, q)\}. \tag{7.5}$$

The first minus sign in (7.5) is dictated by the GSO projection but the second is a matter of taste and has been chosen so that

$$\chi^{\mathrm{L}}_{\mathrm{TM}}(x, q)=q\chi_{\mathrm{B}}(x, q)q^{-3/2}\chi_{D_{12}}(x, q) \tag{7.6}$$

where, as in (6.5),

$$\chi_{D_{12}}(x, q)=\prod_{n=1}^{\infty}(1-q^n)^{-12}\sum_{m\in\Gamma_{12}} x^m q^{(m\cdot m)/2} \tag{7.7}$$

and Γ_{12} is an odd lattice defined by

$$\Gamma_{12}=\{m\,|\,2m_i\in\mathbb{Z},\ m_i-m_j\in\mathbb{Z},\ (|m|-1)/2\in\mathbb{Z},\ i,j=1,2,\ldots,12\}. \tag{7.8}$$

Table 9. $q^{-3/2}\chi_{D_{12}}(x, q)$ expressed in terms of SO(24) characters $[\lambda](x)_{24}$ and $[\Delta; \lambda](x)_{24}$.

	q^{-1}	1	q	q^2	q^3	q^4	q^5	Dimension
$[1]$	1	1	2	3	6	9	16	24
$[1^3]$		1	1	3	5	10	16	2 024
$[1^5]$			1	1	3	5	11	42 504
$[1^7]$				1	1	3	5	346 104
$[1^9]$					1	1	3	1 307 504
$[1^{11}]$						1	2	2 496 144
$[21]$			1	2	4	8	15	4 576
$[21^3]$				1	2	5	10	210 496
$[21^5]$					1	2	5	2 841 696
$[21^7]$						1	2	15 997 696
$[21^9]$							1	43 266 496
$[2^21]$					1	2	5	299 000
$[2^21^3]$						1	2	8 288 280
$[2^21^5]$							1	74 989 200
$[2^31]$							1	9 472 320
$[3]$					1	1	3	2 576
$[31^2]$						1	2	388 080
$[31^4]$							1	9 614 000
$[32]$							1	351 624
Number of tensor states at each level	24	2 048	49 152	614 400	5 373 952	37 122 048	216 072 192	
$[\Delta; 0]_-$		1	1	2	4	7	12	2 048
$[\Delta; 1]_+$			1	2	4	8	15	47 104
$[\Delta; 1^2]_-$				1	2	5	10	516 096
$[\Delta; 1^3]_+$					1	2	5	3 579 904
$[\Delta; 1^4]_-$						1	2	17 616 896
$[\Delta; 1^5]_+$							1	65 286 144
$[\Delta; 2]_-$					1	2	5	565 248
$[\Delta; 21]_+$						1	3	8 243 200
$[\Delta; 21^2]_-$							1	64 204 800
Number of spinor states at each level		2 048	49 152	614 400	5 373 952	37 122 048	216 072 192	
Total number of states at each level	24	4 096	98 304	1 228 800	10 747 904	74 244 096	432 144 384	

It then follows that

$$\chi_{D_{12}}(\boldsymbol{x}, q) = \phi(q)^{-12}\tfrac{1}{2}\{-\theta_1(\boldsymbol{\xi}, \tau) + \theta_2(\boldsymbol{\xi}, \tau) + \theta_3(\boldsymbol{\xi}, \tau) - \theta_4(\boldsymbol{\xi}, \tau)\} \tag{7.9}$$

$$= \tfrac{1}{2}\Bigg(\prod_{n=1}^{\infty}\prod_{l=1}^{24}(1 + x_l q^{n-1/2}) - \prod_{n=1}^{\infty}\prod_{l=1}^{24}(1 - x_l q^{n-1/2})$$

$$+ q^{3/2}\Delta(\boldsymbol{x})_{24}\prod_{n=1}^{\infty}\prod_{l=1}^{24}(1 + x_l q^{n}) - q^{3/2}\Delta'(\boldsymbol{x})_{24}\prod_{n=1}^{\infty}\prod_{l=1}^{24}(1 - x_l q^{n})\Bigg) \tag{7.10}$$

$$= \sum_{\substack{\mu \\ |\mu|\,\mathrm{odd}}} \{\mu'\}(q')_{\infty} D(q)_{\infty}[\mu](\boldsymbol{x})_{24} + q^{3/2}\sum_{\mu}\{\mu'\}(q)_{\infty}F(q)_{\infty}[\Delta;\mu]_{(-1)^{|\mu|-1}}(\boldsymbol{x})_{24} \tag{7.11}$$

as in (6.12)-(6.14). This expansion yields for the first few levels the spectrum displayed in table 9. Apart from the tachyonic lowest level the spectrum is remarkable for the equality between the number of bosonic and fermionic states at each level. This supersymmetry owes its origin to the cube of the Jacobi identity (5.9) which leads, via (6.25*b*), to the relationship between the $k=12$ GSO-projected Neveu–Schwarz and Ramond partition functions first pointed out by Thierry-Mieg [14]. The left-hand sector of the giant superstring model is then obtained by deleting the lowest tachyonic level of the expansion given in table 9 and forming a product with q times the expansion of table 1. The right-hand sector is given precisely by the expansion of table 1. The complete spectrum is then recovered in the usual way by matching the levels of the left- and right-hand sectors.

It is clear that the techniques described here have application to a wide variety of string models and we anticipate that they can be developed further to encompass newer twisted versions of heterotic strings [29].

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